YAU COLLEGE MATH CONTESTS TEAM ALGEBRA 2018

Problem 1

Let d_i $(1 \leq i \leq n)$ be positive integers such that $\sum_{i=1}^{n} \frac{1}{d_i} > 1$. For a prime number p, let $\mathbf{F}_p$ be the finite field of p elements. For

$$f(x_1, \cdots, x_n) = x_1^{d_1} + x_2^{d_2} + \cdots + x_n^{d_n},$$

prove that the number

$$N := \#\{(x_1, \cdots, x_n) \in \mathbf{F}_p^n | f(x_1, \cdots, x_n) = 0\}$$

is divisible by p. Hint: consider the sum $\sum_{(x_1,\cdots,x_n)\in\mathbf{F}_p^n} f(x_1, \cdots, x_n)^{p-1}$.

Problem 2

Let G be a finite group acting on the polynomial ring $R = \mathbf{k}[x_1, \cdots, x_n]$ with n variables $x_1, \cdots, x_n$. Let $S := \{f \in R \mid g \cdot f = f, \ \forall g \in G\}$ be the subring of invariants. Prove that S is a finitely generated $\mathbf{k}$-algebra.

Problem 3

Let k be any field. Let $R = k[[t]]$ be the ring of formal power series over k. Let M be a finitely generated free R-module. Let $v_1, \cdots, v_n \in M$, and denote their images in M/tM by $\bar{v}_1, \cdots, \bar{v}_n$. Assume that $\{\bar{v}_1, \cdots, \bar{v}_n\}$ is a basis of the vector space M/tM over k. Prove that $\{v_1, \cdots, v_n\}$ is an R-basis of the module M.

Remark. This is a special case of Nakayama's lemma.

Problem 4

Let $\mu(t) \in k[t]$ be the minimal polynomial of $A \in M_n(k)$ and

$$W_A = \{ \ AX - XA \mid \forall \ X \in M_n(k) \} \subset M_n(k).$$

Prove that

$$\dim(W_A) \leq n^2 - \deg(\mu(t))$$

and the equality holds if and only if $\deg(\mu(t)) = n$.